Math 304 (Spring 2015) - Homework 7

Problem 1.

Find the angle between the vectors $\vec{x} = (1,1,1)$ and $\vec{y} = (1,0,1)$. (You can express your answer in terms of $\arccos\theta$.)

Solution: Let θ be the angle between $\vec{x}$ and $\vec{y}$. Then

$$\cos\theta = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|} = \frac{2}{\sqrt{6}}.$$

So $\theta = \arccos(\frac{2}{\sqrt{6}})$.

Problem 2.

Let V be the subspace of $\mathbb{R}^3$ spanned by $v_1 = (1,0,1)$ and $v_2 = (1,0,0)$. Find a basis of $V^\perp$.

Solution: $V^\perp$ consists of vectors (a,b,c) which satisfy the following conditions.

$$\begin{cases} (1,0,1)\cdot(a,b,c) = 0 \\ (1,0,0)\cdot(a,b,c) = 0 \end{cases}$$

That is,

$$\begin{cases} a + c = 0 \\ a = 0 \end{cases}$$

So we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ s \\ 0 \end{pmatrix}$$

It follows that

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

forms a basis of $V^\perp$.

Problem 3.

Suppose W is a subspace of $\mathbb{R}^n$. Show that $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Solution: To show that $W^\perp$ is a subspace of $\mathbb{R}^n$, we need to show that $W^\perp$ is nonempty, and is closed under addition and scalar multiplication.

(i) The zero vector 0 is certainly in $W^\perp$. So $W^\perp$ is nonempty.

(ii) If $w_1, w_2 \in W^\perp$, that is, $w_1 \cdot v = 0$ and $w_2 \cdot v = 0$ for all $v \in W$, then

$$(w_1 + w_2) \cdot v = w_1 \cdot v + w_2 \cdot v = 0 + 0 = 0.$$

This implies that $w_1 + w_2 \in W^\perp$.

(iii) If $w_1 \in W^\perp$, and $\alpha \in \mathbb{R}$, then

$$(\alpha w_1) \cdot v = \alpha(w_1 \cdot v) = 0$$

for all $v \in W^\perp$. This implies $\alpha w_1 \in W^\perp$.

Problem 4.

Let $v = (1, 2, -2, 0)$ and $w = (2, 0, 3, 1)$ in $\mathbb{R}^4$.

(a) Find the scalar projection of v onto w.

(b) Find the vector projection of v onto w.

Solution:

(a) The scalar projection of v onto w is given by

$$\frac{v \cdot w}{\|w\|} = \frac{-4}{\sqrt{14}}$$

(b) The vector projection of v onto w is given by

$$\left(\frac{v \cdot w}{\|w\|^2}\right) w = \frac{-2}{7}(2, 0, 3, 1) = (\frac{-4}{7}, 0, \frac{-6}{7}, \frac{-2}{7})$$

Problem 5.

(a) Find the distance between the point $(2, 3, 4)$ and the plane

$$x + y + z = 3$$

(b) Recall that we use three equations to describe a line in $\mathbb{R}^3$. For example a line that passes through the point $(1, -1, 5)$ with the direction $\vec{v} = (2, 3, 4)$ is given by

$$x = 2t + 1, \quad y = 3t - 1, \quad z = 4t + 5.$$

Now given two lines

$$L_1 : x = t + 1,\ y = 3t + 1,\ z = 2t - 1,$$

and

$$L_2 : x = 2t - 2,\ y = 2t + 3,\ z = t + 1,$$

suppose a plane H is parallel to both L_1 and L_2. Moreover, H passes through the point $(0, 1, 0)$. Find the equation of the plane H.

Solution:

(a) Pick a point on the plane. For example, $(0, 0, 3)$ is a point on the plane. Form the vector that goes from the point $(2, 3, 4)$ to this point $(0, 0, 3)$. This is

$$v = (-2, -3, -1)$$

Now the distance is given by the scalar projection of v onto the normal vector $n = (1, 1, 1)$ of the plane

$$\frac{v \cdot n}{\|n\|} = \frac{-6}{\sqrt{3}} = -2\sqrt{3}$$

In fact, we need to the absolute value of this number, since the distance is always nonnegative. So the distance is

$$2\sqrt{3}$$

(b) To find the equation of the plane, we need to find its normal vector. The normal vector n of the plane is orthogonal to both direction vectors of the lines. Denote the direction vectors of the lines by

$$v_1 = (1, 3, 2), \quad v_2 = (2, 2, 1)$$

So n is the cross product of v_1 and v_2:

$$n = \begin{vmatrix} i & j & k \\ 1 & 3 & 2 \\ 2 & 2 & 1 \end{vmatrix} = (-1, 3, -4)$$

So the equation of the plane is

$$-x + 3y - 4z = d$$

for some d. Plug in the point $(0, 1, 0)$ into this equation to solve for d, and we get

$$-x + 3y - 4z = 3.$$